Available online at www.sciencedirect.com

SciVerse ScienceDirect

Physics Procedia 33 (2012) 1461 – 1467

Physics Procedia

ELSEVIER

2012 International Conference on Medical Physics and Biomedical Engineering

The Separate Parameter Estimation in Univariate Fuzzy Linear Regression Model

Zhiwen Zhao

College of Mathematics,Jilin Normal University,Siping, China
zhaozhiwen@126.com

Abstract

Kim et al.[1] proposed a new fuzzy linear regression model and studied the asymptotic normality and strong consistency of the fuzzy least squares estimation. In this paper, we proposed a new estimation method to estimate the parameters in univariate fuzzy linear regression. Compared with the traditional least squares , our method is robust.

Keywords: triangular fuzzy number, the least squares estimation, fuzzy linear regression model, unbiased estimation

1 Introduction

Consider the following fuzzy linear regression model:

$$Y_i = \beta_0 \oplus \beta_1 X_i \oplus \Phi_i, \tag{1}$$

where $X_i = (x_i, \xi_i^l, \xi_i^r)_\Delta$ and $Y_i = (y_i, \eta_i^l, \eta_i^r)_\Delta$, $(i = 1, \cdots, n)$ are fuzzy random variable, β_0 and β_1 are unknown parameters and $\Phi_i = (\varepsilon_i, \varepsilon_i - \theta_i^l, \theta_i^r - \varepsilon_i)_\Delta$ are fuzzy error terms. Model (1) was proposed by Kim et al. [1]. When the model element are crisp, model (1) reduces to the classical linear regression model. Furthermore, in model (1), two kinds of uncertainty, vagueness and randomness, are considered.

Fuzzy regression was first proposed by Tanaka (see [2]). Using a suitable notion of expectation and variance for fuzzy random variable, Näther[3] developed a type of linear estimation theory with random fuzzy data. The statistical inference problem with fuzzy data are condidered by various authors(see Aytar [4], Zadeh [5], Hong and Hwang [6] and Sava and Mursaleen [7] et al.) . Kim et al. [1] extended the standard linear regression model to include specific cases where the observations are vague. When $p = 1$,

1875-3892 © 2012 Published by Elsevier B.V. Selection and/or peer review under responsibility of ICMPBE International Committee.
Open access under CC BY-NC-ND license. doi:10.1016/j.phpro.2012.05.239

they obtained the least square estimation of β_0 and β_1. In this paper, we proposed a new estimation method-separate parameter estimation to estimate the parameters in model (1).

In what follows, we introduce some basic theory about fuzzy mathematics.

We denoted a triangular fuzzy number by $A = (m, l, r)_\Delta$. The set of all triangular fuzzy number is denoted as $T(R^1)$. A linear structure is defined on $T(R^1)$ by

$$(m_1, l_1, r_1)_\Delta \oplus (m_2, l_2, r_2)_\Delta = (m_1 + m_2, l_1 + l_2, r_1 + r_2)_\Delta,$$

$$\lambda(m, l, r)_\Delta = \begin{cases} (\lambda m, \lambda l, \lambda r)_\Delta, & \lambda > 0, \\ (\lambda m, -\lambda r, -\lambda l)_\Delta, & \lambda < 0, \\ (0,0,0)_\Delta, & \lambda = 0. \end{cases}$$

Throughout this paper, we use the following metric defined by Kim et al.[1]:
For $X = (m_1, l_1, r_1)_\Delta$ and $Y = (m_2, l_2, r_2)_\Delta$, define

$$d_H^2(X, Y) = \left\{ \frac{(m_2 - l_2) + m_2 + (m_2 + r_2)}{3} - \frac{(m_1 - l_1) + m_1 + (m_1 + r_1)}{3} \right\}^2.$$

This paper is organized as follows: In Section II, we derive the separate parameter estimator of univariate fuzzy linear regression model. The estimation characters are presented in Section III.

2 The separate parameter estimation

In order to obtain the separate parameter estimation of model (1), we first prove some lemmas.
Lemma 1: If $X = (m_1, l_1, r_1)_\Delta$ and $Y = (m_2, l_2, r_2)_\Delta$,
then we have
(i) $\alpha(-Y) = (-\alpha)Y$,
(ii) $\alpha(X \oplus Y) = (\alpha X) \oplus (\alpha Y)$.
Proof. (i) Observe that

$$\alpha(-Y) = \begin{cases} (-\alpha m_2, \alpha r_2, \alpha l_2)_\Delta, & \alpha \geq 0, \\ (-\alpha m_2, -\alpha l_2, -\alpha r_2)_\Delta & \alpha < 0 \end{cases} \text{and}$$

$$(-\alpha)Y = \begin{cases} (-\alpha m_2, \alpha r_2, \alpha l_2)_\Delta, & \alpha \geq 0, \\ (-\alpha m_2, -\alpha l_2, -\alpha r_2)_\Delta & \alpha < 0, \end{cases}$$

which prove Lemma 1.
ii) By using the similar method used in the proof of Lemma 1, we can prove (ii), so we omit here.
Lemma 2: If $X = (m_1, l_1, r_1)_\Delta$ and $Y = (m_2, l_2, r_2)_\Delta$, then we have $(\alpha X) \oplus (-\alpha Y) = \alpha(X \oplus (-Y))$.

By Lemma 1 and the definition of addition and number multiplication, we can prove Lemma 2.

Statistical analysis under (1) usually involves estimation of the parameter β_0 and β_1. Suppose that the observed data are X_i and $Y_i, i = 1, \cdots, n$. Let

$$Y_i = (y_i, \eta_i^l, \eta_i^r)_\Delta,$$
$$X_i = (x_i, \xi_i^l, \xi_i^r)_\Delta,$$
$$\Delta Y_i = Y_i \oplus (-Y_{i-1}),$$

$$\Delta X_i = X_i \oplus (-X_{i-1})$$

and

$$\Delta \Phi_i = \Phi_i \oplus (-\Phi_{i-1}).$$

we have

$$\Delta Y_i = (y_i - y_{i-1}, \eta_i^l + \eta_{i-1}^r, \eta_i^r + \eta_{i-1}^l)_\Delta$$

and

$\beta_0 \oplus \beta_1 X_i \oplus \Phi_i \oplus (-\beta_0 \oplus \beta_1 X_{i-1} \oplus \Phi_{i-1}) = \beta_1 \Delta X_i \oplus \Delta \Phi_i$.

Note that

$\beta_1 \geq 0$: $\beta_1 \Delta X_i = (\beta_1 (x_i - x_{i-1}), \beta_1 (\xi_i^l + \xi_{i-1}^r), \beta_1 (\xi_i^r + \xi_{i-1}^l))_\Delta$

and

$\beta_1 < 0$: $\beta_1 \Delta X_i = (\beta_1 (x_i - x_{i-1}), -\beta_1 (\xi_i^r + \xi_{i-1}^l), -\beta_1 (\xi_i^l + \xi_{i-1}^r))_\Delta$.

The optimal solution is obtained by minimizing , in the least squares sense,

$$Q(\beta_1) = \sum_{i=2}^{n} d_H^2 (\Delta Y_i, \beta_1 \Delta X_i) = \sum_{i=2}^{n} ((y_i - y_{i-1} - \beta_1 (x_i - x_{i-1}) + \frac{\eta_i^r + \eta_{i-1}^l - (\eta_i^l + \eta_{i-1}^r)}{3} - \frac{\beta_1 (\xi_i^r + \xi_{i-1}^l - (\xi_i^l + \xi_{i-1}^r))}{3})^2 .$$

Let

$$T_i = y_i - y_{i-1} + \frac{\eta_i^r + \eta_{i-1}^l - (\eta_i^l + \eta_{i-1}^r)}{3}$$

and

$$W_i = x_i - x_{i-1} + \frac{\xi_i^r + \xi_{i-1}^l - (\xi_i^l + \xi_{i-1}^r)}{3} .$$

We have

$$Q(\beta_1) = \sum_{i=2}^{n} (T_i - \beta_1 W_i)^2 .$$

It is easy to show that its minimum is attained at the point $\hat{\beta}_1 = \dfrac{\sum_{i=2}^{n} T_i W_i}{\sum_{i=2}^{n} W_i^2}$.

In what follows, we consider the estimation of β_0.

Let $Q(\beta_0) = \sum_{i=2}^{n} d_H^2 (Y_i, \beta_0 + \hat{\beta}_1 X_i)$. Note that if $\hat{\beta}_1 \geq 0$,

then we have

$$\hat{\beta}_1 X_i = (\hat{\beta}_1 x_i, \hat{\beta}_1 \xi_i^l, \hat{\beta}_1 \xi_i^r)_\Delta$$

and

$$\beta_0 + \hat{\beta}_1 X_i = (\beta_0 + \hat{\beta}_1 x_i, \hat{\beta}_1 \xi_i^l, \hat{\beta}_1 \xi_i^r)_\Delta .$$

Further, if $\hat{\beta}_1 < 0$, then we have

$$\hat{\beta}_1 X_i = (\hat{\beta}_1 x_i, -\hat{\beta}_1 \xi_i^r, -\hat{\beta}_1 \xi_i^l)_\Delta$$

and

$$\beta_0 + \hat{\beta}_1 X_i = (\beta_0 + \hat{\beta}_1 x_i, -\hat{\beta}_1 \xi_i^r, -\hat{\beta}_1 \xi_i^l)_\Delta .$$

Therefore,

$$Q(\beta_0) = \sum_{i=1}^{n} d_H^2(Y_i, \beta_0 + \hat{\beta}_1 X_i) = \sum_{i=1}^{n} (\beta_0 + \hat{\beta}_1 x_i - y_i + \frac{\hat{\beta}_1(\xi_i^r - \xi_i^l)}{3} - \frac{\eta_i^r - \eta_i^l}{3})^2.$$

By simple algebra calculation, we obtain

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n}(y_i + \frac{1}{3}(\eta_i^r - \eta_i^l)) - \hat{\beta}_1(\frac{1}{n}\sum_{i=1}^{n}(x_i + \frac{\xi_i^r - \xi_i^l}{3})).$$

3 The character of estimate

In this section, we consider the character of $\hat{\beta}_0$ and $\hat{\beta}_1$.

Similar to the discussion of Kim et al. [1], we assume that $X_{ij}, j = 0,1,\cdots,p; i = 1,\cdots,n$ are known fixed design points. In order to obtain our results, the following conditions are made:

(C1) ε_i are i.i.d. r.v.'s with $E[\varepsilon_i] = 0$ and $Var[\varepsilon_i] = \sigma_\varepsilon^2 (< \infty)$.

(C2) θ_i^r, θ_i^l are i.i.d. r.v.'s with $E[\theta_i^r] = 0$, $E[\theta_i^l] = 0$ and $Var[\theta_i^r] = \sigma_r^2 (< \infty)$, $Var[\theta_i^l] = \sigma_l^2 (< \infty)$.

(C3) ε_i, θ_i^r and θ_i^l are mutually uncorrelated.

Remark 1. The above conditions (C1)-(C3) are used by kim et at.[1].

Now we are ready to state our main results.

Theorem 1: Under the conditions (C1)-(C3), we have $\hat{\beta}_0$ and $\hat{\beta}_1$ are the unbiased estimation of β_0 and β_1 respectively.

Proof. First, we prove that

$$E(\hat{\beta}_1) = \beta_1.$$

Note that

$$\beta_0 \oplus \beta_1 X_i \oplus \Phi_i = \begin{cases} (\beta_0 + \beta_1 x_i + \varepsilon_i, \beta_1\xi_i^l + \varepsilon_i - \theta_i^l, \beta_1\xi_i^r - \varepsilon_i + \theta_i^r)_\Delta, & \beta_1 \ge 0, \\ (\beta_0 + \beta_1 x_i + \varepsilon_i, -\beta_1\xi_i^r + \varepsilon_i - \theta_i^l, -\beta_1\xi_i^l - \varepsilon_i + \theta_i^r)_\Delta, & \beta_1 < 0. \end{cases}$$

and

$$Y_i = (y_i, \eta_i^l, \eta_i^r)_\Delta = \beta_0 \oplus \beta_1 X_i \oplus \Phi_i.$$

We have

$$\begin{aligned} T_i = y_i - y_{i-1} + \frac{\eta_i^r + \eta_{i-1}^l - (\eta_i^l + \eta_{i-1}^r)}{3} &= \beta_1\, (x_i - x_{i-1} + \frac{\xi_i^r + \xi_{i-1}^l - (\xi_i^l + \xi_{i-1}^r)}{3}) + \frac{\varepsilon_i - \varepsilon_{i-1}}{3} \\ &+ \frac{\theta_i^r + \theta_i^l}{3} - \frac{\theta_{i-1}^r + \theta_{i-1}^l}{3} \\ &= \beta_1\, W_i + \frac{\varepsilon_i - \varepsilon_{i-1}}{3} + \frac{\theta_i^r + \theta_i^l}{3} - \frac{\theta_{i-1}^r + \theta_{i-1}^l}{3}. \end{aligned}$$

Therefore, we obtain

$$\hat{\beta}_1 = \frac{\sum_{i=2}^{n} T_i W_i}{\sum_{i=2}^{n} W_i^2} = \frac{\sum_{i=2}^{n}(\beta_1 W_i + \frac{\varepsilon_i - \varepsilon_{i-1}}{3} + \frac{\theta_i^r + \theta_i^l}{3} - \frac{\theta_{i-1}^r + \theta_{i-1}^l}{3})W_i}{\sum_{i=2}^{n} W_i^2}$$

$$= \beta_1 + \frac{\sum_{i=2}^{n}(\frac{\varepsilon_i - \varepsilon_{i-1}}{3} + \frac{\theta_i^r + \theta_i^l}{3} - \frac{\theta_{i-1}^r + \theta_{i-1}^l}{3})W_i}{\sum_{i=2}^{n} W_i^2}.$$

By conditions (C1)-(C3), It is easy to know that

$$E\left(\hat{\beta}_1\right) = 0,$$

which implies (i). We complete the proof of (i).
Next we prove that

$$E\left(\hat{\beta}_0\right) = \beta_0.$$

Observe that

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n}(y_i + \frac{1}{3}(\eta_i^r - \eta_i^l)) - \hat{\beta}_1(\frac{1}{n}\sum_{i=1}^{n}(x_i + \frac{\xi_i^r - \xi_i^l}{3})).$$

By

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

and

$$\eta_i^r - \eta_i^l = \beta_1(\xi_i^r - \xi_i^l) + \theta_i^r - \varepsilon_i - (\varepsilon_i - \theta_i^l),$$

we have

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i + \varepsilon_i + \frac{1}{3}(\xi_i^r - \xi_i^l + \theta_i^r - 2\varepsilon_i + \theta_i^l)) \quad - \hat{\beta}_1(\frac{1}{n}\sum_{i=1}^{n}(x_i + \frac{\xi_i^r - \xi_i^l}{3})).$$

Hence, we have

$$E(\hat{\beta}_0) = \frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i + \frac{1}{3}(\xi_i^r - \xi_i^l)) \ - \beta_1(\frac{1}{n}\sum_{i=1}^{n}(x_i + \frac{\xi_i^r - \xi_i^l}{3})) = \beta_0,$$

which implies that $\hat{\beta}_0$ is the unbiased estimation of β_0.

The proof of Theorem 1 is completed.

Theorem 2: $\hat{\beta}_0$ and $\hat{\beta}_1$ are the linear combination of

$$y_i + \frac{\eta_i^r - \eta_i^l}{3}.$$

Proof. First we prove that $\hat{\beta}_1$ are the linear combination of

$$y_i + \frac{\eta_i^r - \eta_i^l}{3}.$$

Note that

$$\hat{\beta}_1 = \frac{\sum_{i=2}^{n} T_i W_i}{\sum_{i=2}^{n} W_i^2}.$$

$$=\frac{\sum_{i=2}^{n}(y_i-y_{i-1}+\frac{\eta_i^r+\eta_{i-1}^l-(\eta_i^l+\eta_{i-1}^r)}{3})W_i}{\sum_{i=2}^{n}W_i^2}$$

$$=\frac{\sum_{i=2}^{n}(y_i+\frac{\eta_i^r-\eta_i^l}{3}-(y_{i-1}+\frac{\eta_{i-1}^l-\eta_{i-1}^r}{3}))W_i}{\sum_{i=2}^{n}W_i^2}$$

$$=\frac{-W_2}{\sum_{i=2}^{n}W_i^2}(y_1+\frac{\eta_1^r-\eta_1^l}{3})+\frac{W_2-W_3}{\sum_{i=2}^{n}W_i^2}(y_2+\frac{\eta_2^r-\eta_2^l}{3})$$

$$+\cdots+\frac{W_{n-1}-W_n}{\sum_{i=2}^{n}W_i^2}(y_{n-1}+\frac{\eta_{n-1}^r-\eta_{n-1}^l}{3})$$

$$+\frac{W_n}{\sum_{i=2}^{n}W_i^2}(y_n+\frac{\eta_n^r-\eta_n^l}{3}).$$

That is, $\hat{\beta}_1$ are the linear combination of $y_i+\frac{\eta_i^r-\eta_i^l}{3}$.

Next we prove that $\hat{\beta}_0$ are the linear combination of

$$y_i+\frac{\eta_i^r-\eta_i^l}{3}.$$

By

$$\hat{\beta}_0=\frac{1}{n}\sum_{i=1}^{n}(y_i+\frac{1}{3}(\eta_i^r-\eta_i^l))-\hat{\beta}_1(\frac{1}{n}\sum_{i=1}^{n}(x_i+\frac{\xi_i^r-\xi_i^l}{3}))$$

and

$$\hat{\beta}_1=\frac{-W_2}{\sum_{i=2}^{n}W_i^2}(y_1+\frac{\eta_1^r-\eta_1^l}{3})+\frac{W_2-W_3}{\sum_{i=2}^{n}W_i^2}(y_2+\frac{\eta_2^r-\eta_2^l}{3})$$

$$+\cdots+\frac{W_{n-1}-W_n}{\sum_{i=2}^{n}W_i^2}(y_{n-1}+\frac{\eta_{n-1}^r-\eta_{n-1}^l}{3})$$

$$+\frac{W_n}{\sum_{i=2}^{n}W_i^2}(y_n+\frac{\eta_n^r-\eta_n^l}{3}),$$

we have

$$\hat{\beta}_0=(\frac{W_2}{\sum_{i=2}^{n}W_i^2}\frac{1}{n}\sum_{i=1}^{n}(x_i+\frac{\xi_i^r-\xi_i^l}{3})+\frac{1}{n})(y_1+\frac{\eta_1^r-\eta_1^l}{3})\quad-(\frac{W_2-W_3}{\sum_{i=2}^{n}W_i^2}\frac{1}{n}\sum_{i=1}^{n}(x_i+\frac{\xi_i^r-\xi_i^l}{3})+\frac{1}{n})(y_2+\frac{\eta_2^r-\eta_2^l}{3})$$

$$+\cdots$$

$$-(\frac{W_{n-1}-W_n}{\sum_{i=2}^{n}W_i^2}\frac{1}{n}\sum_{i=1}^{n}(x_i+\frac{\xi_i^r-\xi_i^l}{3})+\frac{1}{n})(y_{n-1}+\frac{\eta_{n-1}^r-\eta_{n-1}^l}{3})\quad -(\frac{W_n}{\sum_{i=2}^{n}W_i^2}\frac{1}{n}\sum_{i=1}^{n}(x_i+\frac{\xi_i^r-\xi_i^l}{3})+\frac{1}{n})(y_n+\frac{\eta_n^r-\eta_n^l}{3}),$$

which implies that $\hat{\beta}_0$ are the linear combination of

$$y_i+\frac{\eta_i^r-\eta_i^l}{3}.$$

Theorem 2 is proved.

4 Conclusion

In this paper, we obtained the estimate of the parameters in univariate fuzzy linear regression model. It is not difficult to find that if we use crisp data instead of fuzzy observations, our estimates reduce to the classical estimates.

References

[1] H. K. Kim , J. H. Yoon and Y. Li, "Asymptotic properties of least squares estimation with fuzzy observations," Information Sciences, vol. 178, pp. 439-451 , 2008.

[2] H. Tanaka ,"Fuzzy data analysis by possibilistic linear models," Fuzzy sets and systems, vol. 24, pp. 363-375,1987.

[3] W. Näther, "Linear statistical inference for fuzzy data," Statistics. vol. 29, pp. 221-240, 1997.

[4] S. Aytar, "Statistical limit points of sequences of fuzzy numbers," Information Sciences, vol. 165, pp.129-138, 2004.

[5] L. A. Zadeh ,"Toward a generalized theory of uncertainty (GTU)-an outline," Information Sciences, vol. 172, pp.1-40, 2005.

[6] D. H. Hong and C. Hwang, "Extended fuzzy regression models using regularization method," Information Sciences, vol. 164, pp. 31−46, 2004.

[7] E. Sava and Mursaleen., "On statistically convergent double sequences of fuzzy numbers," Information Sciences, vol. 162, pp.183−192, 2004.